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Deformation quantization, quantization and the Klein–Gordon equation

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Abstract
The aim of this paper is to give a basic introduction to deformation quantization (DQ) to physicists. We compare DQ to canonical quantization and path integral methods. It is described how certain issues such as the roles of associativity, coordinate invariance, dynamics and operator orderings are understood in the context of DQ. Convergence issues in DQ are mentioned. Additionally, we formulate the Klein–Gordon (KG) equation in DQ. Original results are discussed which include the exact construction of the Fedosov star-product on the dS and AdS spacetimes. Also, the KG equation is written down for these spacetimes.

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1. Introduction

In this paper, we discuss several issues regarding quantization and how some of them can be better understood by using deformation quantization (DQ). These issues include the role of coordinate independence and associativity in canonical quantization, and the role of the Lagrangian in the path integral method. Another issue discussed is the operator ordering ambiguities in quantization in the context of DQ

In addition, we illustrate how to write down the Klein–Gordon (KG) equation in DQ, and how to move back and forth from Hilbert space representations to DQ. It is verified, for the case of dS and AdS, that this KG equation and algebra of observables yield the standard results (see, for example, Frønsdal (1965), (1973), (1975a), (1975b)).

The main problem in DQ, as I see it, is related to the standard treatments of deformation products which rely heavily on series expansions in a formal parameter $\hbar$. To partially address the convergence of these series, the Fedosov star-product (a generalization of the Groenewold–Moyal star-product) is computed exactly for the examples of the dS and AdS spacetimes. Once the star-algebra is computed, the Klein–Gordon (KG) equation is then calculated.

2. Quantization on spacetimes

This section is a brief summary of some important issues (which can be confusing) about how to properly construct quantum theories on spacetimes using canonical quantization, path integral methods and DQ. This is an attempt to ascertain some of the essential features of quantum theories. We begin with canonical quantization formulated by Dirac.

2.1. Canonical quantization

The Dirac canonical quantization map Q is a map that tries to assign to each phase-space function f an operator $Q(f)$ (also denoted by $\hat{f}$) that acts on an appropriate Hilbert space. Q is defined by the following four properties:

(1) $Q(c_1 f + c_2 g) = c_1 Q(f) + c_2 Q(g)$,
(2) $Q(\{f, g\}) = [Q(f), Q(g)]/\mathrm{i}\hbar$,
(3) $Q(1) = I$ and
(4) $Q(x)$, $Q(p)$ are represented irreducibly,

for all constants $c_1, c_2 \in \mathbb{C}$, $\{,\}$ is the Poisson bracket, and where I is the unit element in the algebra.

However, there is a major problem with the above setup. The theorem of Groenewold and van Howe states that a consistent quantum theory satisfying rules 1 through 4 is impossible.

It can be easily seen that property 2 is inconsistent by trying to quantize the function $9x^2p^2$ in two ways. One using $9x^2p^2 = \{x^3, p^3\}$ and the other using $9x^2p^2 = \{\sqrt{3}x^2p, \sqrt{3}xp^2\}$. You will see that you obtain two different values for $Q(9x^2p^2)$ which is a contradiction.

Standard canonical quantization gets around this 'no go' thereom by running the procedure for functions that are at most quadratic in the phase-space variables x and p (see Giulini (2003)). The resulting elements $\{Q(x), Q(p), Q(x^2), Q(p^2), Q(xp)\}$ are forced to form the basis of an associative operator algebra which becomes our observable algebra. The procedure Q, subsequently, is consistent *only* on this subset. Therefore, standard canonical quantization is understood in these terms, by the quantization of these basic variables (x, p). The main problem is that the procedure seems to depend on which coordinates (x, p) you choose.

There are ways to get around this problem by modifying the properties above other than running the procedure on at most quadratic variables (x, p). DQ solves the inconsistency by modifying property 2. This is achieved by forcing associativity of the resulting algebra of observables (see Gozzi and Reuter (1994)). Another way of fixing this problem is by abandoning property 4 which is the approach adopted by prequantization in geometric quantization. To go from prequantization to full quantization in general is an unsolved problem in geometric quantization (see Woodhouse (1980)).

The main reason for abandoning property 2 is that it is inconsistent with associativity. First, we start with definition.

Definition. *A Poisson algebra is any algebra equipped on phase space with a product $C(\cdot,\cdot)$ where the antisymmetric part of C for any functions f and g is the Poisson bracket:*

$$C(f, g) - C(g, f) = [f, g]_P.$$

The identity element is 1:

$$C(1, f) = C(f, 1) = f.$$

An example of a Poisson algebra is $C(f, g) = 1 + \frac{1}{2}[f, g]_P$.

A Poisson algebra is necessarily non-associative and so it is simply a matter of apples and oranges. On the one hand you have the non-associative Poisson Algebra (our apples) and on the other hand you have the associative algebra of observables (our oranges). Q then tries to map apples to oranges and it seems obvious that there will be inconsistencies in this mapping. If you run Q only on polynomials that are at most quadratic in x and p then associativity issues never need to come up. However, Q hides the coordinate invariance of the observable algebra that should result from the original Poisson algebra.

2.2. Path integral methods

Feynman rewrote the time-dependent Schrödinger equation as a path integral. The advantage of rewriting it in this way is that it is manifestly coordinate independent. It is based on an S-matrix which concentrates the focus on how states evolve, i.e., the propagator $\langle x_f t_f | x_i t_i \rangle$, where $x_i := x(t_i)$ and $x_f := x(t_f)$. Starting with a Lagrangian $L(x, \dot{x})$ the propagator can be written

$$\langle x_f t_f | x_i t_i \rangle = N \int Dx \exp\left[\frac{1}{\mathrm{i}\hbar}\int_{t_i}^{t_f} \mathrm{d}t\, L(x, \dot{x})\right].$$

Here the sum over all paths is denoted by $\int Dx$ and N is the normalization constant.

Witten (1988) showed that the path integrals on an arbitrary four-dimensional manifold of a twisted supersymmetric QFT are topological invariants called Donaldson's polynomial invariants. Thus his model as well as others like it are diffeomorphism invariant and regarded as topological field theories because the Hilbert spaces (in a BRST sense) are global topological objects. The only observables here are those of topological invariants. This established that the method of path integral quantization is generally covariant and a major reason of its huge success. The only problem here is that the path integral is just a calculational tool in quantum theory for the Schrödinger evolution of states. For example, it has not been solved how you could get the spectra of the hydrogen atom from this approach. In itself, the path integral is not a full quantum theory.

2.3. Deformation quantization

So far we are left with two not-so-appealing options: choose between a quantization that depends on the coordinates (canonical quantization) or just use the Schrödinger evolution of states (the path integral). This brings us to deformation quantization (DQ).

In Groenewold (1946) (and later in Moyal (1949)) realized that the Weyl quantization procedure $\mathcal{W}$ along with Wigner's inverse map $\mathcal{W}^{-1}$ could be used to create an associative, noncommutative product of the two functions f and g of phase-space variables defined by $f * g := \mathcal{W}^{-1}(\mathcal{W}(f)\mathcal{W}(g))$ which has the familiar commutators:

$$[x^\mu, p_\nu]_* = \mathrm{i}\hbar\delta^\mu_\nu, \qquad [x^\mu, x^\nu]_* = 0 = [p_\mu, p_\nu]_*$$

where:

$$f * g = f \exp\left[\frac{\mathrm{i}\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\overrightarrow{\partial}}{\partial x^\mu}\right)\right] g$$

and the arrows denote the direction in which the derivative acts.

In a coordinate independent formulation we have:

$$f * g = f \exp[\overleftrightarrow{P}]g \qquad \overleftrightarrow{P} := \overleftarrow{\partial}_A \frac{\mathrm{i}\hbar}{2}\omega_{AB}\overrightarrow{\partial}_B \tag{1}$$

where $\overleftrightarrow{P}$ is the Poisson bracket and ∂_A is a (flat) torsion-free phase-space connection $(\partial \otimes \omega = 0)$.

In summary, what was obtained was another equivalent formulation of the quantum theory on phase-space, that we call deformation quantization (DQ). DQ is valid for all phase-space functions and not just ones which are at most quadratic in position and momenta (see Hancock *et al* (2004)). Moreover, this is a diffeomorphism covariant quantization which does not depend on the choice of dynamics (like the Lagrangian in the path integral).

3. Operator ordering ambiguities

The Weyl quantization map $\mathcal{W}$ on flat spacetimes corresponds to a symmetric ordering quantization, e.g.

$$\mathcal{W}(xp) = \tfrac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \qquad \mathcal{W}(x^2 p) = \tfrac{1}{3}(\hat{x}^2\hat{p} + \hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2).$$

A different ordering choice would correspond to a different quantization procedure $\mathcal{W}_\lambda$ and, in an analogous way, we define a star-product by (see Hirshfeld and Henselder 2002):

$$f *_\lambda g := \mathcal{W}_\lambda^{-1}(\mathcal{W}_\lambda(f)\mathcal{W}_\lambda(g)).$$

An example of another ordering is standard ordering $\mathcal{W}_\lambda(xp) := \hat{x}\hat{p}$ which corresponds to the standard star-product $*_S$. In some choice of coordinates (x, p) it is:

$$f *_S g = f \exp\left[\mathrm{i}\hbar \frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu}\right] g. \tag{2}$$

Here we observe that different operator orderings correspond to different star-products.

Now we have a remarkable theorem:

Theorem. *All star-products on a symplectic manifold (a generalized phase-space) fall into equivalence classes which are parametrized by a formal series in* $\hbar$ *with coefficients in the second de Rham cohomology group* $H^2_{dR}[[\hbar]]$.[1]

However, we are not sure of the physical consequences of these mathematical equivalences except that in general the algebra of observables are isomorphic, however each spectra of observables may be different (see Hirshfeld and Henselder 2002). These equivalences do tell us that the algebra of observables for any two star-products are isomorphic.

4. The Klein–Gordon equation and the Fedosov star-product

In order to gain a basic feel for DQ, we will recast the well known equation Klein–Gordon (KG) equation into DQ. In this section we will sometimes implicitly use $\mathcal{W}$ and $\mathcal{W}^{-1}$ to go from Hilbert spaces to phase-space (and back). For more details of the arguments below, see Tillman and Sparling (2006a, 2006b).

The KG equation is obtained by promoting the classical Minkowskian relativistic invariant $p_\mu p^\mu - m^2$ to a Hilbert space operator.

States of definite mass $|\phi_m\rangle$ are then solutions to the eigenvalue equation:

$$(\hat{p}_\mu \hat{p}^\mu - m^2)|\phi_m\rangle = 0, \qquad \langle\phi_m|\phi_m\rangle = 1$$

This is the KG equation because in x-space we have:

$$(\partial_\mu \partial^\mu + m^2/\hbar^2)\phi_m(x) = 0.$$

[1] This theorem is due to the contribution of many people (see Dito and Sternheimer (2002) for a brief history of the classification).

To reformulate states as quantities in phase-space (i.e. in DQ) we use Wigner's inverse map $\mathcal{W}^{-1}$:

$$\rho_m := \mathcal{W}^{-1}(|\phi_m\rangle\langle\phi_m|).$$

The functions ρ_m are known as Wigner functions.

The KG equation on Minkowski space in DQ can be now written as:

$$H * \rho_m = \rho_m * H = m^2 \rho_m$$
$$H = p_\mu * p^\mu$$
$$Tr_*(\rho_m) = 1, \qquad \bar{\rho}_m = \rho_m$$

where $*$ is the Groenewold-Moyal star-product and Tr_* is the trace over all degrees of freedom.

In an analogous derivation (and by adding an arbitrary Ricci term) we can formulate the KG equation on an arbitrary spacetime:

$$H * \rho_m = \rho_m * H = m^2 \rho_m \tag{3}$$

$$\begin{aligned} H &= p_\mu * p^\mu + \xi R(x) \\ Tr_*(\rho_m) &= 1, \qquad \bar{\rho}_m = \rho_m \end{aligned} \tag{4}$$

where $*$ is now the Fedosov star-product (a generalization of the Groenewold-Moyal star-product), $g_{\mu\nu}(x)$ is a configuration space metric, $R(x)$ is the Ricci scalar associated to this metric, $p^\mu := g^{\mu\nu} p_\nu$, and $\xi \in \mathbb{C}$ is an arbitrary constant.

The properties of the Fedosov star are (see Fedosov (1996), Tillman and Sparling (2006a, 2006b)):

(1) it is diffeomorphism covariant;
(2) it can be constructed on all symplectic manifolds (including all phase-spaces) perturbatively in powers of $\hbar$;
(3) it assumes no dynamics (e.g. Hamiltonian or Lagrangian), symmetries, or even a metric;
(4) the limit $\hbar \to 0$ yields classical mechanics, and
(5) it is equivalent to an operator formalism by a Weyl-like quantization map σ^{-1}.

The Fedosov star-product is given by an iterative construction, and, with convergence issues aside, all star products on any symplectic manifold are formally equivalent to a Fedosov star (see Dito and Sternheimer (2002)). We add that the role played $\mathcal{W}$ (and $\mathcal{W}^{-1}$) is the flat section in Weyl bundle (called σ^{-1} in Fedosov (1996)) over the symplectic manifold.

4.1. The dS and AdS space-times

We constructed the Fedosov star-product for the phase-space a class of constant curvature manifolds in Tillman and Sparling (2006a). The following is a summary of these results for the cases of the dS and AdS spacetimes.

One of the goals of these results is to obtain a nonperturbative construction of the Fedosov star-product for the dS/AdS spacetimes. Another is to verify that the algebra of observables and the KG equation reproduced previous results of Frønsdal (1965, 1973, 1975a, 1975b).

We first embed dS/AdS in a flat five dimensional space given by the embedding formulas:

$$\eta_{\mu\nu} x^\mu x^\nu = 1/C \qquad \text{and} \qquad x^\mu p_\mu = A$$

where C and A are some real arbitrary constants, and η is the embedding flat metric. For dS $\eta = \mathrm{diag}(1, -1, -1, -1, -1)$, $C < 0$ and AdS $\eta = \mathrm{diag}(1, 1, -1, -1, -1)$, $C > 0$.

For brevity we omit the technical details of the calculations and simply give the results. We obtain the exact results for the Fedosov star-commutators:

$$[x^{\mu}, x^{\nu}]_* = 0[x_{\mu}, M_{\nu\rho}]_* = \mathrm{i}\hbar x_{[\nu}\eta_{\rho]\mu} \qquad [M_{\mu\nu}, M_{\rho\sigma}]_* = \mathrm{i}\hbar(M_{\rho[\mu}\eta_{\nu]\sigma} - M_{\sigma[\mu}\eta_{\nu]\rho}) \tag{5}$$

indices run from 0 to 4, $M_{\mu\nu} = x_{[\mu} * p_{\nu]}$, $x_{\mu} = \eta_{\mu\nu}x^{\nu}$.

The conditions of the embedding $x^{\mu}x_{\mu}$, $x^{\mu}p_{\mu}$ become the Casimir invariants of the algebra in group theoretic language.

We now summarize our two key observations:

(1) M's generate $\mathbb{SO}(1, 4)$ and $\mathbb{SO}(2, 3)$ for dS and AdS respectively.
(2) M's and x's generate $\mathbb{SO}(2, 4)$ for *both* dS and AdS.

By calculating $R = -16C$ and $p_{\mu} * p^{\mu}$ in terms of M and x the Hamiltonian (4) is:

$$H = 2CM_{\mu\nu} * M^{\mu\nu} + (A - 4\mathrm{i}\hbar)AC - 16\xi C \tag{6}$$

where $M_{\mu\nu} * M^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}(1, 4)$ or $\mathbb{SO}(2, 3)$ for dS or AdS respectively.

In the more familiar form of Hilbert space language the KG equation (3) takes the form:

$$(2C\hat{M}_{\mu\nu}\hat{M}^{\mu\nu} + \chi C)|\phi_m\rangle = m^2|\phi_m\rangle \tag{7}$$

where $\langle\phi_m|\phi_m\rangle = 1$, $\mathbb{C} \ni \chi = (A - 4\mathrm{i}\hbar)A - 16\xi$ is an arbitrary constant, and we regard all groups to be in a standard irreducible representation on the set of linear Hilbert space operators.

These subgroups are the symmetry groups of the manifolds for dS or AdS respectively. Again, $\hat{M}_{\mu\nu}\hat{M}^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}(1, 4)$ or $\mathbb{SO}(2, 3)$ for dS or AdS respectively. Therefore, the above KG equation (7) states that the eigenstates of mass $|\phi_m\rangle$ label the different representations of $\mathbb{SO}(1, 4)$ and $\mathbb{SO}(2, 3)$ for dS and AdS respectively sitting inside the full group of observables $\mathbb{SO}(2, 4)$ which is confirmed by Frønsdal (1965, 1973) as well as others.

5. Conclusions

As we saw, the main advantage over both canonical quantization and path integral methods is that DQ is both coordinate independent and independent of the dynamics (e.g. Lagrangian). Also, quantization procedures of different operator ordering correspond, in the context of DQ, to different star products. However, the physical meaning of equivalent star products is still under investigation.

It is shown how to conceptually move from a Hilbert space formalism to DQ and back using implicitly the map $\mathcal{W}$ and its generalization σ^{-1}. This helped us reformulate quantities and equations, like the KG equation, from DQ into Hilbert spaces (and back). For the specific cases of dS and AdS spacetimes the Fedosov star-product was calculated and the results obtained were the expected ones (see Frønsdal (1965), (1973), (1975a), (1975b)). However, the fundamental issue of convergence of all formal series in DQ still remains unknown.

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